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FINE STRUCTURE OF BOUNDARY FLOWS IN MEDIA WITH DIFFUSION AND HEAT CONDUCTION

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The influence of the boundary conditions at the surfaces confining multicomponent stratified media on the formation of flows in problems of multicomponent diffusion and thermoconcentration convection is investigated. Exact solutions of these problems are given. Analysis of these solutions shows that several boundary layers (concentration and velocity layers) are formed in the case of multicomponent diffusion, which leads to decomposition of the physical fields and splitting of the characteristic spatial scales. In the case of thermoconcentration convection, a more complicated dynamic structure is formed, which, besides boundary layers, includes injection fronts. The latter have a significant effect on the flow characteristics at distances far exceeding the thickness of the boundary layers.

The appearance of [1] has increased interest in the analysis of boundary effects in stratified and rotating media. A model of steady mountain winds in stably stratified media was constructed by Prandtl [2]. Similar effects are observed on the continental slopes of the ocean [3, 4]. The solutions constructed by Prandtl [2], Phillips [3], and Wunsch [4] describe boundary flow with a unified scale of variation of all physical variables.

The results of [2–4] are not consistent with exact solutions of the problem of diffusion near a vertical or horizontal wall. The asymptotic solution of the nonsteady problem (the approximation of short times) describes more complicated boundary flow with separate scales of variation of velocity and density [5]. Slow Couette flow in a sloping channel demonstrates similar properties [6].

An exact solution of the nonsteady problem with separate scales of variation of velocity and density was constructed by Kistovich and Chashechkin [7–9]. The resulting flow was investigated experimentally by Phillips et al. [10].

Under the actual conditions of a multicomponent medium, one observes stratification due to multicomponent diffusion or thermoconcentration convection [11], whose dynamics is affected by boundary flows [12].

Processes in an isothermal stratified medium with several dissolved substances are called multicomponent diffusion (MCD) below, and the heat propagation in a liquid that is stably stratified in concentration (salinity) is called the thermoconcentration convection (TCC).

Formulation of the Problem. We consider the flow of a stratified multicomponent liquid over an infinite inclined wall. The stratification is given by the distribution of the concentration of admixtures, for which the wall is impermeable, or of one admixture and temperature. We consider the two-dimensional problem in the Boussinesq approximation [13]:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla p}{\rho_0} + \nu\Delta\mathbf{u} + \rho'\mathbf{g}, \qquad \rho' = \beta S \pm \alpha T, \qquad \nabla\cdot\mathbf{u} = 0, \qquad \frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla\bar{S} = k_S\Delta S,$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla\bar{T} = k_T\Delta T, \qquad \bar{S} = S + S_0(z), \qquad \bar{T} = T + T_0(z), \tag{1}$$

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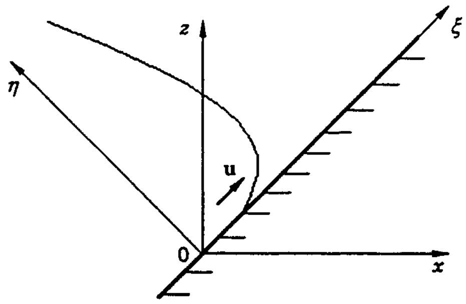

Fig. 1

$$S_0(z) = S_{00}\left(1 - \frac{z}{\Lambda_S}\right), \qquad T_0(z) = T_{00}\left(1 \mp \frac{z}{\Lambda_T}\right).$$

Here $\mathbf{u}$ is the velocity field, p is the pressure minus the hydrostatic pressure, S and T are perturbations in the concentrations of the first and second admixtures in the case of MCD and in the salt concentration and temperature in the case of TCC, $S_0(z)$ and $T_0(z)$ are the stratified distributions of the corresponding physical fields, with the upper signs in the equations for the case of MCD and the lower signs for the case of TCC, Λ_S and Λ_T are the scales of the stratifications, β and α are the coefficients of admixture compression (MCD) or salt compression and thermal expansion (TCC), k_S, k_T, and ν are the diffusivities of the components S and T and kinematic viscosity, respectively, ρ_0 is the density of the pure liquid, and $\mathbf{g}$ is the acceleration of gravity.

The initial conditions are the equality to zero of the perturbations of all the physical fields. The conditions at infinity are the same.

The boundary conditions for the MCD and TCC cases differ. For MCD, we set the normal components of the gradients of the total fields $\bar{S}$ and $\bar{T}$ to zero at the wall. In the TCC case, only the normal component of the salt distribution gradient $\bar{S}$ is set to zero at the wall, and for the temperature we adopt the condition of heat exchange between the wall and the medium, characterized by the heat-transfer coefficient γ. The boundary conditions for the velocity field $\mathbf{u}$ are the attachment condition.

In the general situation, the wall is inclined to the horizontal at an angle α, and, hence, it is convenient to convert to a coordinate system ξ, η attached to the wall (see Fig. 1).

General Method of Solving the Problem. As in [7], the solution is sought in the form of Fourier series in the angle α:

$$u = \sum_{n=1}^{\infty} U_n \sin(n\alpha), \quad S = S_0 + \sum_{n=1}^{\infty} S_n \cos(n\alpha), \quad T = T_0 + \sum_{n=1}^{\infty} T_n \cos(n\alpha), \tag{2}$$

where u is the component of the velocity field along the ξ axis, and the component along the η axis is identically equal to zero.

Before substituting (2) into (1), we first make the variables dimensionless using the rules

$$t^* = \frac{t}{N}, \quad \eta^* = \left(\frac{N}{\nu}\right)^{1/2}\eta, \quad \varepsilon_S = \frac{k_S}{\nu}, \quad \varepsilon_T = \frac{k_T}{\nu}, \quad S^* = \beta S, \quad T^* = \mp\alpha T,$$
$$U_n^* = -2(\Lambda g)^{-1/2}U_n, \quad N = \left(\frac{g}{\Lambda}\right)^{1/2}, \quad \Lambda = \frac{\Lambda_S\Lambda_T}{\Lambda_S + \Lambda_T}, \quad G_n^* = S_n^* \pm T_n^*, \tag{3}$$

and if either of the scales Λ_S or Λ_T equals zero, the other scale is chosen for Λ. The upper sign is chosen in the MCD case and the lower sign in the TCC case.

Substituting (2) into (1) with allowance for (3) generates an infinite system of equations for determining the Fourier coefficients in expansions (2):

$$\frac{\partial U_1}{\partial t} - U_1'' = 2G_0 - G_2,$$

$$\frac{\partial U_n}{\partial t} - U_n'' = G_{n-1} - G_{n+1}, \quad n = 2,\ 3, \ldots,$$
$$\frac{\partial S_n}{\partial t} - \varepsilon_S S_n'' = a_S(U_{n+1} - U_{n-1}), \qquad U_0 = U_{-1} = 0, \tag{4}$$
$$\frac{\partial T_n}{\partial t} - \varepsilon_T T_n'' = a_T\left(U_{n+1} - U_{n-1}\right), \quad n = 0,\ 1, \ldots,$$
where $a_S = \Lambda/4\Lambda_S$ and $a_T = \Lambda/4\Lambda_T$.

As in [8], the solution of system (4) is hindered by the fact that the equations are "engaged" with each other. The iterative procedure for finding the coefficients $\{U_i\}$, $\{S_i\}$, and $\{T_i\}$ is preceded by a change of variables. To derive rules for carrying out such changes, here, as in [7], we found Lie group generators corresponding to system (4), compiled different infinite linear combinations of generators that produce the change of the variables $\{U_i\}$, $\{S_i\}$, and $\{T_i\}$, and, as in [8], performed summation of infinite diverging series [14] that determine the coefficients of the variables of the problem. As a result, we derived rules for changing the variables in (4), which ensures the procedure for solving the problem.

A consistent description of this method does not seem possible, and we therefore give another method for deriving the rules of changing the variables, which was obtained using group analysis.

In the first step of the transformation to the new variables $\{V_i\}$, $\{Q_i\}$, and $\{R_i\}$, we can use the known solution for $\alpha = 0$ [9]. From expansions (2) we obtain
$$T(\alpha = 0) = T_0 + T_1 + T_2 + \ldots, \qquad S(\alpha = 0) = S_0 + S_1 + S_2 + \ldots.$$
The quantities $T(0)$ and $S(0)$ are chosen as the new variables R_1 and Q_1, for which the expansions in $\{T_i\}$ and $\{S_i\}$ are written in somewhat altered form:
$$R_1 = T_0 + T_2 + T_4 + \ldots + T_1 + T_3 + T_5 + \ldots, \quad Q_1 = S_0 + S_2 + S_4 + \ldots + S_1 + S_3 + S_5 + \ldots.$$

In the second step, we need to combine the equations of system (4) so that the right side contains $Q_1 - R_1$. This can be done by introducing the variable
$$V_1 = U_1 + 3U_3 + 5U_5 + 7U_7 + \ldots + 2(U_2 + 2U_4 + 3U_6 + 4U_8 + \ldots).$$

In the third step, we must return to the equations for $\{Q_i\}$ and $\{R_i\}$ and combine them so that the right side contains V_1. This is achieved by introducing the variable
$$R_2 = T_2 + 4T_4 + 9T_6 + 16T_8 + \ldots + 2(T_3 + 3T_5 + 6T_7 + 10T_9 + \ldots).$$

Performing this operation by turns with Eqs. (4), we obtain the following system of new variables:
$$\begin{gathered} R_1 = T_0 + T_2 + T_4 + \ldots + T_1 + T_3 + T_5 + \ldots, \\ R_2 = T_2 + 4T_4 + 9T_6 + 16T_8 + \ldots + 2(T_3 + 3T_5 + 6T_7 + \ldots), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ V_1 = U_1 + 3U_3 + 5U_5 + 7U_7 + \ldots + 2(U_2 + 2U_4 + 3U_6 + \ldots), \\ V_2 = U_3 + 5U_5 + 14U_7 + 30U_9 + \ldots + 2(U_4 + 4U_6 + 10U_8 + \ldots), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{gathered} \tag{5}$$
The equations for $\{Q_i\}$ are similar to those for $\{R_i\}$.

Equations (5) are inverted using the known solution for $\alpha = \pi/2$ [9], $T(\alpha = \pi/2) = T_0 - T_2 + T_4 - T_6 + \ldots$, and the rules
$$\begin{gathered} T_0 = R_1^{(e)} - R_2^{(e)} + 3R_3^{(e)} - 10R_4^{(e)} + 35R_5^{(e)} - \ldots, \\ T_2 = R_2^{(e)} - 4R_3^{(e)} + 15R_4^{(e)} - 56R_5^{(e)} + \ldots, \qquad T_4 = \ldots, \\ U_1 = V_1^{(o)} - 3V_2^{(o)} + 10V_3^{(o)} - 35V_4^{(o)} + \ldots, \\ U_3 = V_2^{(o)} - 5V_3^{(o)} + 21V_4^{(o)} - \ldots, \qquad U_5 = \ldots; \end{gathered} \tag{6a}$$

$$T_1 = R_1^{(o)} - R_2^{(o)} + 2R_3^{(o)} - 5R_4^{(o)} + 14R_5^{(o)} - \dots,$$

$$T_3 = R_2^{(o)} - 3R_3^{(o)} + 9R_4^{(o)} - 28R_5^{(o)} + \dots, \qquad T_5 = \dots, \tag{6b}$$

$$U_2 = V_1^{(e)} - 2V_2^{(e)} + 5V_3^{(e)} - 14V_4^{(e)} + \dots, \qquad U_4 = V_2^{(e)} - 4V_3^{(e)} + 14V_4^{(e)} - \dots, \qquad U_6 = \dots.$$

Substituting (5) into (4) yields two systems of equations connecting the Fourier coefficients $\{V_i^{(e)}\}$, $\{Q_i^{(o)}\}$, and $\{R_i^{(o)}\}$ and $\{V_i^{(o)}\}$, $\{Q_i^{(e)}\}$, and $\{R_i^{(e)}\}$:

$$\frac{\partial V_n^{(e)}}{\partial t} - V_n^{(e)''} = R_n^{(o)} - Q_n^{(o)}, \quad n = 1,\ 2, \dots,$$

$$\frac{\partial Q_1^{(o)}}{\partial t} - \varepsilon_S Q_1^{(o)''} = 0, \quad \frac{\partial Q_n^{(o)}}{\partial t} - \varepsilon_S Q_n^{(o)''} = -a_S V_{n-1}^{(e)}, \tag{7a}$$

$$\frac{\partial R_1^{(o)}}{\partial t} - \varepsilon_T R_1^{(o)''} = 0, \quad \frac{\partial R_n^{(o)}}{\partial t} - \varepsilon_T R_n^{(o)''} = -a_T V_{n-1}^{(e)}, \quad n = 2,\ 3, \dots;$$

$$\frac{\partial V_1^{(o)}}{\partial t} - V_1^{(o)''} = 2\left(R_1^{(e)} - Q_1^{(e)}\right), \quad \frac{\partial V_n^{(o)}}{\partial t} - V_n^{(o)''} = R_n^{(e)} - Q_n^{(e)},$$

$$\frac{\partial Q_1^{(e)}}{\partial t} - \varepsilon_S Q_1^{(e)''} = 0, \quad \frac{\partial Q_n^{(e)}}{\partial t} - \varepsilon_S Q_n^{(e)''} = -a_S V_{n-1}^{(o)}, \tag{7b}$$

$$\frac{\partial R_1^{(e)}}{\partial t} - \varepsilon_T R_1^{(e)''} = 0, \quad \frac{\partial R_n^{(e)}}{\partial t} - \varepsilon_T R_n^{(e)''} = -a_T V_{n-1}^{(o)}, \quad n = 2,\ 3, \dots.$$

Solving Eqs. (7) enables one to determine the coefficients of expansions (5) and, using the inversion equations, to determine the coefficients of expansions (2), i.e., to solve the stated problems.

Because of differences in the boundary conditions for the MCD and TCC cases, the problems are solved separately below.

Exact Solution of the MCD Problem. The boundary conditions for the MCD problem are given in the form of the condition of impermeability of the wall to the admixtures,

$$\left.\frac{\partial S}{\partial \eta}\right|_{\eta=0} = \left.\frac{\partial T}{\partial \eta}\right|_{\eta=0} = 0$$

and the attachment condition, $u|_{\eta=0} = 0$, which, after transformation to the variables $\{V_i\}$, $\{Q_i\}$, and $\{R_i\}$, take the form

$$\left.V_n^{(e)}\right|_{\eta=0} = 0, \quad n = 1, 2, \dots, \quad \left.Q_n^{(o)'}\right|_{\eta=0} = \left.R_n^{(o)'}\right|_{\eta=0} = 0, \quad n = 2,\ 3, \dots,$$

$$\left.Q_1^{(o)'}\right|_{\eta=0} = A^{(o)}, \quad \left.R_1^{(o)'}\right|_{\eta=0} = B^{(o)}, \quad n = 2,\ 3, \dots, \tag{8}$$

$$\left.V_n^{(o)}\right|_{\eta=0} = \left.Q_n^{(e)'}\right|_{\eta=0} = \left.R_n^{(e)'}\right|_{\eta=0} = 0, \quad n = 1,\ 2, \dots,$$

where $A^{(o)}$ and $B^{(o)}$ are constants.

Introducing, as in [8], the variable $y = \eta t^{-1/2}$, which is both one of the group invariants of system (4) and a standard substitution in boundary-layer problems, we can represent the solution of problem (7) and (8) in the form

$$Q_n^{(o)} = \tilde{Q}_n^{(o)} t^{(4n-3)/2}, \quad R_n^{(o)} = \tilde{R}_n^{(o)} t^{(4n-3)/2}, \quad V_n^{(e)} = \tilde{V}_n^{(e)} t^{(4n-1)/2},$$

$$Q_n^{(e)} = 0, \ \tilde{Q}_n^{(o)} = \sum_{m=1}^{n} K_m^n U\left(w - \frac{1}{2}, \frac{y}{(2\varepsilon_S)^{1/2}}\right) + \sum_{m=1}^{n-1} L_m^n U\left(w - \frac{1}{2}, \frac{y}{2^{1/2}}\right) + \sum_{m=1}^{n-1} M_m^n U\left(w - \frac{1}{2}, \frac{y}{(2\varepsilon_T)^{1/2}}\right),$$

$$R_n^{(e)} = 0, \ \tilde{R}_n^{(o)} = \sum_{m=1}^{n} D_m^n U\left(w - \frac{1}{2}, \frac{y}{(2\varepsilon_T)^{1/2}}\right) + \sum_{m=1}^{n-1} E_m^n U\left(w - \frac{1}{2}, \frac{y}{2^{1/2}}\right) + \sum_{m=1}^{n-1} F_m^n U\left(w - \frac{1}{2}, \frac{y}{(2\varepsilon_S)^{1/2}}\right), \tag{9}$$

$$V_n^{(o)} = 0,\quad \tilde V_n^{(e)} = \sum_{m=1}^{n} I_m^n U\left(w+\frac{3}{2}, \frac{y}{2^{1/2}}\right) + \sum_{m=1}^{n} J_m^n U\left(w+\frac{3}{2}, \frac{y}{(2\varepsilon_S)^{1/2}}\right) + \sum_{m=1}^{n} C_m^n U\left(w+\frac{3}{2}, \frac{y}{(2\varepsilon_T)^{1/2}}\right),$$

$$w = 4n - 2m, \qquad U(n+1/2, x) = \frac{1}{n}\int_x^\infty (\xi - x)^n \exp(-\xi^2/2)\, d\xi.$$

The coefficients K_m^n, L_m^n, M_m^n, D_m^n, E_m^n, F_m^n, I_m^n, J_m^n, and C_m^n are connected by recursive equations that are similar to the equations [8]

$$K_1^1 = \frac{2A^{(o)}}{\sqrt{\pi}}, \qquad D_1^1 = \frac{2B^{(o)}}{\sqrt{\pi}}, \qquad mK_{m+1}^n = -a_S J_m^{n-1}, \qquad mL_{m+1}^n + \frac{1-\varepsilon_S}{2} L_m^n = -a_S I_m^{n-1},$$

$$mD_{m+1}^n = -a_T C_m^{n-1}, \qquad mE_{m+1}^n + \frac{1-\varepsilon_T}{2} E_m^n = -a_T I_m^{n-1},$$

$$\frac{\varepsilon_S - 1}{2\varepsilon_S} J_n^n = -K_n^n, \qquad mM_{m+1}^n + \frac{1-\varepsilon_S/\varepsilon_T}{2} M_m^n = -a_S C_m^{n-1},$$

$$\frac{\varepsilon_T - 1}{2\varepsilon_T} C_n^n = D_n^n, \qquad mF_{m+1}^n + \frac{1-\varepsilon_T/\varepsilon_S}{2} F_m^n = -a_T J_m^{n-1},$$

$$mI_{m+1}^n = E_m^n - L_m^n, \qquad mJ_{m+1}^n + \frac{\varepsilon_S - 1}{2\varepsilon_S} J_m^n = F_m^n - K_m^n,$$

$$mC_{m+1}^n + \frac{\varepsilon_T - 1}{2\varepsilon_T} C_m^n = D_m^n - M_m^n, \qquad m = 1,\ 2, \ldots, \quad n = 2,\ 3, \ldots\ .$$

From the boundary conditions we obtain the following additional equations:

$$\sum_{m=1}^{n} \frac{2^m (I_m^n + J_m^n + C_m^n)}{\Gamma(2n - m + 3/2)} = 0,$$

$$2^n K_n^n + \sum_{m=1}^{n-1} 2^m \left(K_m^n + \sqrt{\varepsilon_S} L_m^n + \sqrt{\frac{\varepsilon_S}{\varepsilon_T}} M_m^n\right) \frac{\Gamma(n)}{\Gamma(2n-m)} = 0,$$

$$2^n D_n^n + \sum_{m=1}^{n-1} 2^m \left(D_m^n + \sqrt{\varepsilon_T} E_m^n + \sqrt{\frac{\varepsilon_T}{\varepsilon_S}} F_m^n\right) \frac{\Gamma(n)}{\Gamma(2n-m)} = 0,$$

where $\Gamma(x)$ is a gamma function.

In the analysis of solution (9), we consider a typical case where the inequality $\nu > k_S > k_T$ is valid. From (9) and the structure of the function U it follows that the entire space–time can be divided into four domains, for which the characteristic scales are given by the inequalities

$$1)\ \infty > y > \sqrt{2}; \qquad 2)\ \sqrt{2} > y > \sqrt{2\varepsilon_S}; \qquad 3)\ \sqrt{2\varepsilon_S} > y > \sqrt{2\varepsilon_T}; \qquad 4)\ \sqrt{2\varepsilon_T} > y > 0.$$

It is convenient to further analyze the space–time domains by returning to the physical variables η and t. Then, away from the wall, all perturbations are small for $\infty > \eta > \sqrt{2\nu t}$.

In the intermediate domain $\sqrt{2\nu t} > \eta > \sqrt{2k_S t}$, the velocity perturbations reach a maximum, and the density variations are small. This domain can be called a dynamic (or velocity) boundary layer, whose thickness is $\delta_u = O(\sqrt{2\nu t})$.

In the third domain $\sqrt{2k_S t} > \eta > \sqrt{2k_T t}$, the velocity variations are less pronounced than in the second, and the perturbations of the first admixture S reach a maximum whereas the perturbations of the second admixture T are considerably smaller. This domain can be called a first concentration (or density) boundary layer, whose thickness is $\delta_{\rho_1} = O(\sqrt{2k_S t})$.

In the fourth domain $\sqrt{2k_T t} > \eta > 0$, which is adjacent to the inclined wall, the velocity perturbations become even less pronounced, the variations in the concentration of the first admixture decrease, while the variations in the concentration of the second admixture reach a maximum. This domain is called a second concentration boundary layer, whose thickness is $\delta_{\rho_2} = O(\sqrt{2k_T t})$.

Thus, in MCD problems there is a clear separation of the scales at which the most important variations of the different physical fields occur.

Exact Solution of the Problem of Establishment of TCC. In the case of thermoconcentration convection, the boundary conditions change:

$$u\Big|_{\eta=0} = 0, \quad \left\{k_H \frac{\partial H}{\partial \eta} + \gamma_H (H - H_{00})\right\}\Big|_{\eta=0} = 0, \quad H = \begin{cases} S, \\ T \end{cases}$$

(γ_S and γ_T are the coefficients of the salt- and heat-transfer surfaces).

These boundary conditions change in the transformation to the variables $\{V_i\}$, $\{Q_i\}$, and $\{R_i\}$:

$$V_n^{(i)}\Big|_{\eta=0} = 0, \quad n = 1,\ 2, \ldots, \quad Q_n^{(i)\prime} + \gamma_S Q_n^{(i)}\Big|_{\eta=0} = 0, \quad n = 2,\ 3, \ldots,$$
$$Q_1^{(o)\prime} + \gamma_S Q_1^{(o)}\Big|_{\eta=0} = A^{(o)}, \quad Q_1^{(e)\prime} + \gamma_S Q_n^{(e)}\Big|_{\eta=0} = 0, \tag{10}$$
$$R_1^{(i)\prime} + \gamma_T R_1^{(i)}\Big|_{\eta=0} = B^{(i)}, \quad R_n^{(i)\prime} + \gamma_T R_n^{(i)}\Big|_{\eta=0} = 0, \quad n = 2,\ 3, \ldots,$$

where (i) is (e) or (o).

In (10), instead of the condition of salt nonpenetration, we deliberately used the condition of salt exchange with the plane, characterized by the salt-transfer coefficient γ_S. Below, it is shown how the solution of (7) and (10) smoothly becomes the solution of the problem with the salt nonpenetration condition as $\gamma_S \to 0$, and a solution of a more general problem is given.

In the problem considered, it is convenient to introduce new variables, found by a group analysis of Eqs. (7) after reducing boundary conditions (10) to a uniform type:

$$y = \frac{\eta}{2t^{1/2}}, \quad y_S = \frac{\eta}{2(\varepsilon_S t)^{1/2}}, \quad y_T = \frac{\eta}{2(\varepsilon_T t)^{1/2}}, \quad z_S = y_S - \gamma_S(\varepsilon_S t)^{1/2}, \quad z_T = y_T - \gamma_T(\varepsilon_T t)^{1/2}.$$

The solutions of (7) and (10) have the form

$$Q_n^{(o)} = \tilde{Q}_n^{(o)} t^{2(n-1)}, \quad \tilde{Q}_n^{(o)} = \sum_{m=1}^{n} K_m^n I(w-1, y_S, z_S) + \sum_{m=1}^{n-1} L_m^n I(w-1, y_T, z_T) + \sum_{m=1}^{n-1} M_m^n I(w-2, y),$$
$$R_n^{(o)} = \tilde{R}_n^{(o)} t^{2(n-1)}, \quad \tilde{R}_n^{(o)} = \sum_{m=1}^{n} D_m^n I(w-1, y_T, z_T) + \sum_{m=1}^{n-1} E_m^n I(w-1, y_S, z_S) + \sum_{m=1}^{n-1} F_m^n I(w-2, y), \tag{11}$$
$$V_n^{(e)} = \tilde{V}_n^{(e)} t^{2n-1}, \quad \tilde{V}_n^{(e)} = \sum_{m=1}^{n} I_m^n I(w, y) + \sum_{m=1}^{n} J_m^n I(w+1, y_S, z_S) + \sum_{m=1}^{n} C_m^n I(w+1, y_T, z_T),$$

where $w = 4n - 2m$, and the function $I(w, y, z)$ is defined by the equation

$$I(w, y, z) = \exp(-y^2)(\exp(y^2) i^w \operatorname{erfc}(y) + \exp(z^2) i^w \operatorname{erfc}(z)),$$
$$I(w, y) = i^w \operatorname{erfc}(y), \quad i^k \operatorname{erfc}(x) = \int_x^\infty i^{k-1} \operatorname{erfc}(x)\, dx, \qquad k = 0,\ 1,\ 2, \ldots .$$

The function $I(n, y, z)$ introduced above, which can be called a standard integral, is a solution of the differential equation

$$\frac{\partial^2 I}{\partial y^2} + \frac{\partial^2 I}{\partial z^2} + 2y \frac{\partial I}{\partial y} - 2z \frac{\partial I}{\partial z} - 2nI = 0.$$

The structures of the solutions for $V_n^{(o)}$, $Q_n^{(e)}$, and $R_n^{(e)}$ are fully analogous to the structure of (11) and, hence, they are not given. The coefficients K_m^n, L_m^n, M_m^n, D_m^n, E_m^n, F_m^n, I_m^n, J_m^n, and C_m^n of the expansions (11) and expansion coefficients for $V_n^{(o)}$, $Q_n^{(e)}$, and $R_n^{(e)}$ are determined from recursive relations obtained by substituting the solutions into (7) and (10), and they coincide with Eqs. (9) for the MCD problem, except for the equations that follow from the boundary conditions:

$$K_1^1 = \sqrt{\pi\varepsilon_S}A^{(o)}/(\sqrt{\pi} - \gamma_S\sqrt{\varepsilon_S}), \qquad D_1^1 = \sqrt{\pi\varepsilon_T}B^{(o)}/(\sqrt{\pi} - \gamma_T\sqrt{\varepsilon_T}),$$

$$\sum_{m=1}^{n} K_m^n\left(\Phi(\varepsilon_S,0) - \gamma_S\Phi(\varepsilon_T,1/2)\right) + \sum_{m=1}^{n-1} L_m^n\left(\Phi(\varepsilon_T,0) - \gamma_S\Phi(\varepsilon_S,1/2)\right) + \sum_{m=1}^{n-1}\frac{M_m^n}{2^{4n-2m}}\frac{2-\gamma_S}{\Gamma(2n-m)} = 0,$$

$$\sum_{m=1}^{n}\frac{I_m^n}{2^{4n-2m}\Gamma(2n-m+1)} + \sum_{m=1}^{n}\frac{J_m^n + C_m^n}{2^{4n-2m-1}\Gamma(2n-m+3/2)} = 0,$$

$$\sum_{m=1}^{n} D_m^n\left(\Phi(\varepsilon_T,0) - \gamma_T\Phi(1,1/2)\right) + \sum_{m=1}^{n-1} E_m^n\left(\Phi(\varepsilon_T,0) - \gamma_T\Phi(1,1/2)\right) + \sum_{m=1}^{n-1}\frac{F_m^n}{2^{4n-2m}}\frac{2-\gamma_T}{\Gamma(2n-m)} = 0,$$

$$\Phi(\lambda,x) = \frac{2^{2n-4m}}{\sqrt{\lambda}\Gamma(2n-m-x)}.$$

As in the MCD case, the TCC solutions are characterized by separate dynamical, density, and temperature scales. At the same time, two new scales are present in the flow, characterizing the degrees of salt transfer and heat transfer of the plane surface, and the flow dynamics itself differs from that in the MCD case.

Analysis of the Properties of the Solutions Obtained. For a detailed examination of the similarity and difference in flows for the MCD and TCC cases, we introduce the concept of the "front of the boundary layer." The position of this front is determined by the relation $\eta/\sqrt{2kt} = \text{const}_k$, where k is the kinetic coefficient that corresponds to the boundary layer.

The propagation velocity of the front is defined by the equation

$$v_k = \frac{\partial\eta}{\partial t} = \text{const}_k\sqrt{k/2t}. \tag{12}$$

It is seen from (12) that the ratios of the propagation velocities of the fronts of the dynamic and density boundary layers do not change during the entire process of formation and development of the flow. Thus, in the MCD process, the flow is self-similar from the standpoint of the spatial manifestation of the internal scales of the flow.

In the TCC case, in addition to the three fronts indicated above, two fronts, of salt and thermal injection, are present in the flow. If the salt-transfer coefficient of the surface is zero, there is only the thermal-injection front. The positions of the injection fronts are determined by the relations $z_S = \text{const}_S$ and $z_T = \text{const}_T$, from which we can determine the propagation velocities of these fronts:

$$v_H = \text{const}_H\sqrt{k_H/2t} + \sqrt{2}\gamma_H k_H, \quad H = \begin{cases} S, \\ T. \end{cases} \tag{13}$$

It is seen from (12) and (13) that the ratios of the velocities of the injection fronts and the velocities of the boundary-layer fronts are not constants. Moreover, the propagation velocities of the boundary-layer fronts approach zero with time, $\lim_{t\to\infty} v_k = 0$, whereas the velocities of the injection fronts arrive at constant values:

$$\lim_{t\to\infty} v_H = \sqrt{2}\gamma_H k_H, \quad H = \begin{cases} S, \\ T. \end{cases}$$

The time dependence of the distance between the leading boundary of the front and the vertical heater in a stably stratified salt solution was determined in [15]. In the initial stage, the size of the region of the heated liquid grows in proportion to $\sqrt{t}$, which corresponds to the calculated velocity of the front, which is proportional to $\sqrt{1/t}$.

If at the beginning of development of the flow, any of the boundary-layer fronts leads the injection fronts, then it eventually leads all the boundary fronts and changes the relationships among the internal scales of the flow. This phenomenon has been observed experimentally in tests of convection from a heated inclined plane [16].

In contrast to MCD flow, TCC flow is not self-similar, and the distances between the fronts and their relative position vary. At the same time, partial self-similarity of the flow occurs at long times.This means that the relative position of the injection fronts and the relative position of the boundary-layer fronts are preserved. Here TCC flow has a more complicated internal structure, which in addition varies with time.

The differences in structure and dynamics between MCD and TCC flows indicate that direct extension of the results from studies of MCD processes to TCC processes is not justified, especially for developed flows. Partial similarity occurs only at short times, when the injection fronts have just been formed and do not significantly affect the flow structure.

It must be noted that the correctness of the linear approximation in the analysis of convective structures is very limited in the case of thermoconcentration convection, and in actual situations the differences between MCD and TCC processes are even stronger.

Conclusion. Our studies of the influence of boundary conditions showed that in the MCD case, the main feature of the flows is the development of several boundary layers (density and velocity layers), which leads to decomposition of the physical fields of the problem and separation of the characteristic spatial scales. In the TCC case, a more complicated dynamic structure is formed, characterized, in addition to boundary layers, by injection fronts, which have the greatest influence at distances far exceeding the thickness of the boundary layers.

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